

# New shear-free relativistic models with heat flow

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**Abstract** We study shear-free spherically symmetric relativistic models with heat flow. Our analysis is based on Lie's theory of extended groups applied to the governing field equations. In particular, we generate a five-parameter family of transformations which enables us to map existing solutions to new solutions. All known solutions of Einstein equations with heat flow can therefore produce infinite families of new solutions. In addition, we provide two new classes of solutions utilising the Lie infinitesimal generators. These solutions generate an infinite class of solutions given any one of the two unknown metric functions.

**Keywords** Gravitating fluids · Exact solutions · Lie symmetries

## 1 Introduction

In this paper, we consider spherically symmetric radiating spacetimes with vanishing shear which are important in relativistic astrophysics, radiating stars and cosmology. The assumption that the shear vanishes in a spherically symmetric spacetime, in the presence of nonvanishing heat flux, is often made to describe the dynamics of cosmological models. The importance of relativistic heat conducting fluids in modeling inhomogeneous processes, such as galaxy formation and evolution of perturbations, has been pointed out by Krasinski [1]. Some of the early exact solutions in the presence of heat flow were given by Bergmann [2], Maiti [3] and Modak [4]. Deng [5] provided a general method of generating solutions to the Einstein field equations which contains most previously known exact solutions. Heat conducting exact solutions are necessary to generate temperature profiles

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in dissipative processes by integrating the heat transport equation as shown by Trigriner and Pavon [6]. Bulk viscosity with heat flow affects the dynamics of inhomogeneous cosmological models as shown by Deng and Mannheim [7]. Recently Banerjee and Chatterjee [8] and Banerjee *et al* [9] have investigated heat conducting fluids in higher dimensional cosmological models when considering spherical collapse, the appearance of singularities and the formation of horizons. The role of heat flow in gravitational dynamics and perturbations in the framework of brane world cosmological models has been highlighted by Davidson and Gurwich [10] and Maartens and Koyama [11].

The presence of heat flux is necessary for a proper and complete description of radiating relativistic stars. The result of Santos *et al* [12], in one of the first complete relativistic radiating models, indicates that the interior spacetime should contain a nonzero heat flux so that the matching at the boundary to the exterior Vaidya spacetime is possible. Models containing heat flow in astrophysics have been applied to problems in the gravitational collapse, black hole physics, formation of singularities and particle production at the stellar surface in four and higher dimensions. The study of Chang *et al* [13] showed that the process of gravitational collapse of a spherical star with heat flow may serve as a possible energy mechanism for gamma-ray bursts.

Herrera *et al* [14], Maharaj and Govender [15], and Mistry *et al* [16] have shown that relativistic radiating stars are useful in the investigation of the cosmic censorship hypothesis and in describing collapse with vanishing tidal forces. Wagh *et al* [17] presented solutions to the Einstein field equations for a shear-free spherically symmetric spacetime, with radial heat flux by choosing a barotropic equation of state. For particles in geodesic motion a general analytic treatment is possible and solutions are obtainable in terms of elementary and special functions as demonstrated by Thirukkanash and Maharaj [18]. Herrera *et al* [19] found analytical solutions to the field equations for radiating collapsing spheres in the diffusion approximation. These authors demonstrated that the thermal evolution of the collapsing sphere which can be modeled in causal thermodynamics requires heat flow. Note that stellar models with shear are difficult to analyse but particular exact solutions have been found by Naidu *et al* [20] and Rajah and Maharaj [21].

Shear-free fluids are also essential in modeling inhomogeneous cosmological processes. Krasinski [1] points out the need for radiating models in the formation of structure, evolution of voids, the study of singularities and cosmic censorship. Heat conducting fluids are important in cosmological models in higher dimensions and permits collapse without the appearance of an event horizon; this aspect has been studied by Banerjee and Chatterjee [8]. In brane world models the presence of heat flow allows for more general behaviour than in standard general relativity, the analogue of the Oppenheimer-Snyder model of a collapsing dust permits a radiating brane which was proved by Govender and Dadhich [22].

In this paper we intend to analyse the pivotal equation previously studied by Deng [5]. He developed a general (though *ad hoc*) method to generate solutions and obtained a new class of solutions which included various known special cases (see Sect. 2.). We adopt a systematic approach (using Lie theory) to generalise known solutions and generate new solutions of the same equation. The basic features of Lie symmetry analysis are outlined in Sect. 3. In Sect. 4, we extend the Deng [5] known solutions to find new solutions to the fundamental equation utilising Lie

theory. In Sect. 5, we systematically study other group invariant solutions admitted by the fundamental equation. For most of the symmetries, we obtain either an implicit solution or we can reduce the governing equations to a Riccati equation which is difficult to solve in general (though particular solutions can always be found). There are two cases in which we find new exact solutions regardless of the complexity of the generating function chosen. Some brief concluding remarks are made in Sect. 6.

## 2 Radiating Model

Due to the requirements of spherical symmetry and the shear-free condition, the line element can be written as

$$ds^2 = -D^2 dt^2 + \frac{1}{V^2} [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] \quad (1)$$

where  $D$  and  $V$  are functions of  $t$  and  $r$ . In the study of solutions of the Einstein equation with heat flux, Deng [5] studied a shear-free spherically symmetric cosmological model where he considered the energy-momentum tensor as

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} + q_\mu U_\nu + q_\nu U_\mu \quad (2)$$

where  $U_\mu$ ,  $\rho$ ,  $p$  are the four-velocity of the fluid, energy density and pressure, respectively, and  $q_\mu$  is the heat flux. The Einstein field equations are given by

$$\rho = \frac{3V_t^2}{D^2 V^2} + V^2 \left[ \frac{2V_{rr}}{V} - \frac{3V_r^2}{V^2} + \frac{4V_r}{rV} \right] \quad (3a)$$

$$p = \frac{1}{D^2} \left[ \frac{2V_{tt}}{V} - \frac{5V_t^2}{V^2} - \frac{2D_t V_t}{DV} \right] + V^2 \left[ \frac{V_r^2}{V^2} - \frac{2D_r V_r}{DV} + \frac{2D_r}{rD} - \frac{2V_r}{rV} \right] \quad (3b)$$

$$p = \frac{2V_{tt}}{D^2 V} - \frac{5V_t^2}{D^2 V^2} - \frac{2D_t V_t}{D^3 V} + \frac{D_r V^2}{rD} - \frac{VV_r}{r} + \frac{D_{rr} V^2}{D} + V_r^2 - VV_{rr} \quad (3c)$$

$$q = -2V^2 \left[ \frac{V_{tr}}{DV} - \frac{V_t V_r}{DV^2} - \frac{D_r V_t}{D^2 V} \right] \quad (3d)$$

Equations (3b) and (3c) together imply

$$VD_{uu} + 2D_u V_u - DV_{uu} = 0 \quad (4)$$

which is the condition of pressure isotropy with  $u = r^2$ . Glass [23] and Bergmann [2], also discovered that in the comoving system, Einstein field equations generate the pressure isotropy condition given by the equation (4), which is the master equation for the system (3a)-(3d).

A number of authors have obtained various solutions to (4), among which is the conformally flat class

$$D = \frac{c(t)u + d(t)}{a(t)u + b(t)} \quad (5a)$$

$$V = a(t)u + b(t) \quad (5b)$$

where  $a, b, c$ , and  $d$  are arbitrary functions of  $t$ . Sanyal [24] and Modak [4] obtained this class along with other solutions, while Bergmann [2] and Maiti [3] obtained special cases of the class.

A method of generating more general solutions to the master equation (4) has been developed by Deng [5] who found solutions when simple forms of  $V$  or  $D$  are chosen. In finding solutions, Deng [5] considered the master equation as an ordinary differential equation with respect to  $u$ . He treated (4) as a linear equation of  $V$  if  $D$  is a known function and vice versa. His approach was as follows:

- Choose a simple function  $D = D_1$  and find the most general solution  $V = V_1$  of (4).
- Take  $V = V_1$  and find the most general solution  $D = D_2$  obeying (4).
- Take  $D = D_2$  and find the most general solution  $V = V_2$ .

This procedure can be continued indefinitely. By alternating between  $D$  and  $V$ , this process can go on forever generating infinite series of solutions expressed in terms of integrals. This is a powerful method as all known solutions can be generated using this algorithm

In this paper, we show the Deng [5] general method of generating solutions, that gives a general class of solutions which include (5a)-(5b) as special cases, may be extended by a simple invariant transformations. In addition, we reduce the order of (4) via Lie analysis to obtain new solutions not obtainable via the Deng approach.

### 3 Lie analysis

The basic feature of Lie analysis for a system of ordinary differential equations in two dependent variables requires the determination of the one-parameter ( $\varepsilon$ ) Lie group of transformations

$$\bar{u} = f(u, V, D, \varepsilon) \quad (6a)$$

$$\bar{V} = g(u, V, D, \varepsilon) \quad (6b)$$

$$\bar{D} = h(u, V, D, \varepsilon) \quad (6c)$$

that leaves the solution set of the system invariant. It is difficult to calculate these transformations directly. As a result, one tends to look for the infinitesimal form of the transformations, *viz.*

$$\bar{u} = u + \varepsilon \xi(u, V, D) + O(\varepsilon^2) \quad (7a)$$

$$\bar{V} = V + \varepsilon \eta(u, V, D) + O(\varepsilon^2) \quad (7b)$$

$$\bar{D} = D + \varepsilon \zeta(u, V, D) + O(\varepsilon^2). \quad (7c)$$

This form of the transformations can be obtained once we obtain their “generator”

$$G = \xi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial V} + \zeta \frac{\partial}{\partial D} \quad (8)$$

(also called a symmetry of the differential equation) which is a set of vector fields. Having found the symmetries, we can regain the finite (global) form of the transformation (6a)-(6c), by solving

$$\frac{d\bar{u}}{d\varepsilon} = \xi(\bar{u}, \bar{V}, \bar{D}) \quad (9a)$$

$$\frac{d\bar{V}}{d\varepsilon} = \eta(\bar{u}, \bar{V}, \bar{D}) \quad (9b)$$

$$\frac{d\bar{D}}{d\varepsilon} = \zeta(\bar{u}, \bar{V}, \bar{D}) \quad (9c)$$

subject to

$$\bar{u}|_{\varepsilon=0} = u, \quad \bar{V}|_{\varepsilon=0} = V, \quad \bar{D}|_{\varepsilon=0} = D. \quad (10)$$

(The full details can be found in a number of excellent texts such as Bluman and Kumei [25] and Olver [26]).

The determination of the generators is a straight forward process, greatly aided by computer algebra packages (see the treatments of Dimas and Tsoubelis [27] and Cheviakov [28]). We have found the package `PROGRAM LIE` by Head [29] to be the most useful in practice. It is quite remarkable how accomplished such an old package is; it often yields results when its modern counterparts fail.

Utilising `PROGRAM LIE`, we can demonstrate that (4) admits the following Lie point symmetries/vector fields:

$$G_1 = \frac{\partial}{\partial u} \quad (11a)$$

$$G_2 = u \frac{\partial}{\partial u} \quad (11b)$$

$$G_3 = D \frac{\partial}{\partial D} \quad (11c)$$

$$G_4 = V \frac{\partial}{\partial V} \quad (11d)$$

$$G_5 = u^2 \frac{\partial}{\partial u} + uV \frac{\partial}{\partial V} \quad (11e)$$

At this stage, it is usual to use these symmetries to reduce the order of the equation in the hope of finding solutions. Before we proceed with this approach, we show how the finite form of the transformations generated by these symmetries can generate new solutions from known solutions.

#### 4 Extending known solutions

Having found the symmetries of (4) we know that they generate transformations of the form (6a)-(6c) that leaves (4) invariant.

We illustrate the approach using the infinitesimal generator  $G_1$ . First we observe that

$$\xi = 1, \quad \eta = 0, \quad \zeta = 0 \quad (12)$$

We solve equations (9a)-(9c), subject to (10), to obtain

$$\bar{u} = u + a_1 \quad (13a)$$

$$\bar{V} = V \quad (13b)$$

$$\bar{D} = D \quad (13c)$$

This means that (13a)-20c maps equation (4) to the form

$$\bar{V}\bar{D}\bar{u}\bar{u} + 2\bar{D}\bar{u}\bar{V}\bar{u} - \bar{D}\bar{V}\bar{u}\bar{u} = 0. \quad (14)$$

As a result, any existing solution to equation (4) can be transformed to a solution of (14) (and so a solution of (4) itself) by (13a)- 20c. Note that, usually,  $a_1$  is an arbitrary constant. However, since  $V$  and  $D$  depend on  $u$  and  $t$  we take  $a_1$  to be an arbitrary function of time,  $a_1 = a_1(t)$ .

If we now take each of the remaining symmetries successively, we obtain the general transformation as follows:

$$G_1 : \bar{u} = a_1 + u, \quad \bar{D} = D, \quad \bar{V} = V \quad (15a)$$

$$G_2 : \bar{u} = e^{a_2}(a_1 + u), \quad \bar{D} = D, \quad \bar{V} = V \quad (15b)$$

$$G_3 : \bar{u} = e^{a_2}(a_1 + u), \quad \bar{D} = e^{a_3}D, \quad \bar{V} = V \quad (15c)$$

$$G_4 : \bar{u} = e^{a_2}(a_1 + u), \quad \bar{D} = e^{a_3}D, \quad \bar{V} = e^{a_4}V \quad (15d)$$

$$G_5 : \bar{u} = \frac{e^{a_2}(a_1 + u)}{1 - a_5 e^{a_2}(a_1 + u)}, \quad \bar{D} = e^{a_3}D, \quad \bar{V} = \frac{e^{a_4}V}{1 - a_5 e^{a_2}(a_1 + u)} \quad (15e)$$

The combination of all the transformation of symmetries, therefore leads to the general relationship:

$$\bar{u} = \frac{e^{a_2}(a_1 + u)}{1 - a_5 e^{a_2}(a_1 + u)} \quad (16a)$$

$$\bar{D} = e^{a_3}D \quad (16b)$$

$$\bar{V} = \frac{e^{a_4}V}{1 - a_5 e^{a_2}(a_1 + u)} \quad (16c)$$

where the  $a_i$  are all arbitrary function of time.

Thus any known solution of equation (4) can be transformed to a new solution of equation (4) via (16a)-(16c). For example, the particular Deng [5] solution

$$D = 1, \quad V = \alpha(t)u + \beta(t) \quad (17)$$

is transformed to the new solution

$$\bar{u} = \frac{e^{a_2}(a_1 + u)}{1 - a_5 e^{a_2}(a_1 + u)} \quad (18a)$$

$$\bar{V} = \frac{e^{a_4}(\alpha(t)u + \beta(t))}{1 - a_5 e^{a_2}(a_1 + u)} \quad (18b)$$

$$\bar{D} = e^{a_3} \quad (18c)$$

All the solutions in the Deng [5] class, the conformally flat models (5a)-(5b), the result listed in Krasinski [1] and Stephani *et al* [30] can be extended by (16a)-(16c) to produce new solutions of (4). Also note that all the new results that we derive in the next section can be similarly extended via (16a)-(16c).

## 5 New Solutions via Lie symmetries

The usual use of symmetries of ordinary differential equations is to reduce the order of a differential equation. For symmetries (11a)-(11e) we obtain either an implicit solution of (4) or we can reduce the governing equations to a Riccati equation. Both these results are no improvement to that of Deng [5], *i.e.* we need to choose simple forms for one of the functions (either  $D$  or  $V$ ) in order to solve for the other. However there are two cases in which we can find new solutions regardless of the complexity of the function chosen.

### 5.1 The choice $D = D(V)$

An obvious relation to consider is when one dependent variable in (4) is a function of the other. In general such an approach usually results in a more complicated equation. However, using the Lie symmetry  $G_1$  (which gives the same result as  $G_2$ ), we can make significant progress. For our purposes we use the partial set of invariants of

$$G_1 = \frac{\partial}{\partial u} \quad (19)$$

given by

$$p = V \quad (20a)$$

$$q(p) = V_u \quad (20b)$$

$$r(p) = D \quad (20c)$$

This transformation reduces equation (4) to

$$q'(p) [r(p) - pr'(p)] = q(p) [pr''(p) + 2r'(p)] \quad (21)$$

which can be integrated to give

$$q = q_0 e^{\int \frac{2r' + pr''}{r - r'p} dp} \quad (22)$$

Substituting for the metric functions via (20a)-(20c), we can integrate one more time to give the solution

$$\int \left[ e^{-\int \frac{2DV + VD_{VV}}{D - VD_V} dV} \right] dV = q_0 u + u_0 \quad (23)$$

where  $q_0$  and  $u_0$  are arbitrary functions of time. This means that, given any function  $V$  depending on  $D$ , we can work out  $V$  explicitly from (23). Such a relationship between  $V$  and  $D$  has not been found previously. Note that since (4) is linear, once we obtain  $V$  via (23) we can use it to obtain the general solution of (4) using standard techniques for solving linear equations. (In the case of  $G_2$ , we obtain exactly the same solution as (23).)

We illustrate this method with simple examples. Using the particular Deng [5] solution  $D = 1$ , we evaluate (23) to obtain

$$V = q_0(t)u + u_0 \quad (24)$$

which is exactly what Deng [5] obtained.

If we take  $D = V^2$ , then equation (23) is reduced to

$$\int V^6 dV = q_0 u + u_0 \quad (25)$$

and hence

$$V_1 = (\bar{q}_0 u + \bar{u}_0)^{1/7} \quad (26a)$$

$$D = (\bar{q}_0 u + \bar{u}_0)^{2/7} \quad (26b)$$

which is a solution of (4). Having found  $V$  from (23) we can generate a new solution as follows: the second independent solution of (4) has the form

$$V_2 = y (q_0 u + u_0)^{1/7} \quad (27)$$

with  $y$  being unknown. Then (4) becomes

$$y'' - \frac{2q_0}{7(q_0 u + u_0)} y' = 0 \quad (28)$$

with solution

$$y = \frac{C_1}{q_0} (q_0 u + u_0)^{9/7} + C_2 \quad (29)$$

Hence

$$V = \frac{C_1}{q_0} (q_0 u + u_0)^{10/7} + C_2 (q_0 u + u_0)^{1/7} \quad (30)$$

where  $C_1$  and  $C_2$  are arbitrary functions of time, is the general solution to (4) when  $D = (\bar{q}_0 u + \bar{u}_0)^{2/7}$ .

## 5.2 The choice $W = \frac{V}{D}$

We also consider the ratio of  $V$  to  $D$  (and later  $D$  to  $V$ ) to generate a new solution. Incidentally both ratios arise as a result of a combination of the generators  $G_3$  and  $G_4$  given by

$$G_3 + G_4 = D \frac{\partial}{\partial D} + V \frac{\partial}{\partial V} \quad (31)$$

This symmetry combination gives rise to the invariant

$$W = \frac{V}{D} \quad (32)$$

Then equation (4) is transformed by (32) to the form

$$-2W D_u^2 + D^2 W_{uu} = 0 \quad (33)$$

with solution

$$D = C_1(t) \exp \left( \int \pm \frac{\sqrt{W_{uu}}}{\sqrt{2W}} du \right) \quad (34)$$

Given any function  $W = W(u)$  we can integrate the right hand side and find a form for  $D$ . If we take  $W = a(t)u + b(t)$ , then (34) gives

$$D = \bar{C}_1(t) \quad (35)$$



and

$$V = \bar{C}_1(t)(a(t)u + b(t)) \quad (36)$$

which corresponds to a Deng [5] solution.

Alternatively, we could substitute the inverse of (32), *i.e.*

$$\widehat{W} = \frac{D}{V} \quad (37)$$

into (4) and obtain

$$2D'^2\widehat{W}^2 - 2D^2\widehat{W}'^2 + D^2\widehat{W}\widehat{W}'' = 0 \quad (38)$$

with solution

$$D = C_2(t) \exp \left( \pm \int \frac{\sqrt{\widehat{W}'^2 - \frac{1}{2}\widehat{W}\widehat{W}''}}{\widehat{W}} du \right) \quad (39)$$

Again, given any function  $W = W(u)$  we can integrate the right hand side and find a form for  $D$ . If we take  $W = a(t)u + b(t)$  as before, in (39), we find that

$$D_1 = C_2(t)(a(t)u + b(t)) \quad (40)$$

and

$$V_1 = C_2(t) \quad (41)$$

which again is essentially a solution of Deng [5]. However, we also have

$$D_2 = \frac{\bar{C}_2(t)}{a(t)u + b(t)} \quad (42)$$

and

$$V_2 = \frac{\bar{C}_2(t)}{(a(t)u + b(t))^2} \quad (43)$$

satisfies (4), thus obtaining two different solutions from the same seed function.

Observe that (34) and (39) will contain all solutions of Deng [5] for appropriately chosen seed functions  $W$  or  $\widehat{W}$  which are ratios of the metric functions. While Deng's approach requires simply chosen forms of either  $D$  or  $V$  in order to integrate (4), we have no such requirement. We are always able to reduce (4) to the quadratures (34) or (39) regardless of the complexity of the seed functions, a result not obtainable via Deng's approach.

## 6 Conclusion

We have been able to extend Deng's solutions [5] of the Einstein field equations governing shear-free heat conducting fluids in general relativity. This was accomplished by using simple transformations based on the invariance properties of the equation under study *viz.* (4). In addition, motivated by the invariants of the symmetries admitted by (4) we were able to reduce (4) to quadratures for *any* given seed function. This leads to three new classes of solutions for infinite families of functional forms involving  $D(V)$ ,  $W$  and  $\widehat{W}$ . This is an improvement on the approach of Deng [5] who required 'simple' functional forms for  $D$  or  $V$  to be chosen before (4) could be solved. This again promotes the use of Lie's theory of extended groups to analyse the Einstein field equations arising in different applications/models in general relativity.

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